

Fermionic mode entanglement in quantum information

Nicolai Friis^{1,*}, Antony R. Lee^{1,†} and David Edward Bruschi^{1,2,‡}

¹*School of Mathematical Sciences, University of Nottingham,
University Park, Nottingham NG7 2RD, United Kingdom and*

²*School of Electronic and Electrical Engineering, University of Leeds,
Woodhouse Lane, Leeds LS2 9JT, United Kingdom*

(Dated: November 2012)

We analyze fermionic modes as fundamental entities for quantum information processing. To this end we construct a density operator formalism on the underlying Fock space and demonstrate how it can be naturally and unambiguously equipped with a notion of subsystems in the absence of a global tensor product structure. We argue that any apparent similarities between fermionic modes and qubits are superficial and can only be applied in limited situations. In particular, we discuss the ambiguities that arise from different treatments of this subject. Our results are independent of the specific context of the fermionic fields as long as the canonical anti-commutation relations are satisfied, e.g., in relativistic quantum fields, or fermionic trapped ions.

I. INTRODUCTION

Fermionic systems have been analyzed as agents for quantum information processing in a multitude of studies, ranging from discussions of fermionic modes of relativistic quantum fields [1–11], over fermionic lattices [12], and fermionic Gaussian states [13], to discussions of the entanglement between fixed numbers of indistinguishable particles [14–24]. In the latter case, only pure states of fixed particle numbers are considered and a selection of entanglement measures are available, see, e.g., Ref. [20]. However, these restrictions seem to be much more limiting than required. From the point of view of quantum information theory it is natural to ask for an extension to incoherent mixtures of quantum states. Furthermore, from the perspective of a relativistic description particle numbers are not usually conserved, i.e., the particle content of a given pure state is observer dependent [25]. Only the coherent superpositions of pure states of different (electric) charges are forbidden [26].

In the light of this fact it is therefore reasonable to consider the entanglement between fermionic modes, in a similar way as this is conventionally done for bosonic modes, e.g., for Gaussian states [27]. However, due to the Pauli exclusion principle fermionic modes are naturally restricted to two degrees of freedom, i.e., each mode can be unoccupied or contain a single excitation. This has provided many researchers with an ad hoc justification for the comparison with qubits – two level systems used in quantum information. The central question then simply is: *Can fermionic modes be considered as qubits?* The short answer to this question is “No”, but in limited situations certain techniques from

the study of qubits can be applied in fermionic systems. This has incited debates among scientists, see, e.g., the exchange in Refs. [28–30]. In the following we shall give a more precise answer to the question above, along with a detailed description of the problem. The superselection rules further restrict the possible operations that can be performed on single-mode subsystems and it was argued that this should lead to a modified definition of the entanglement between modes [20]. At least for fixed particle content this problem can be circumvent [31]. Moreover, even if quantum correlations are not directly accessible, a transfer of the entanglement to systems that are not encumbered by such restrictions should be possible, thus justifying the use of unmodified measures for mode entanglement.

The main aim of this paper is to establish a clear framework for the implementation of fermionic field modes as vessels for quantum information tasks. To this end we present an analysis of the problem at hand, i.e., how the modes in a fermionic Fock space can be utilized as subsystems for quantum information processing. We present a framework that is based on simple physical requirements in which this can be achieved. We further discuss the issues and restrictions in mapping fermionic modes to qubits and we show how previous work and proposed solutions, e.g., invoking superselection rules [29], fit into this framework.

The article is structured as follows: We start with a brief discussion of the description of fermionic Fock spaces in Sec. II and how density operators are constructed on such spaces in Sec. III. We then go on to formulate the “fermionic ambiguity” that has been pointed out in Ref. [28] in Sec. IV. Subsequently, we reinterpret this as an ambiguity in the definition of mode-subsystems, which can be resolved by physical consistency conditions, in Sec. V. Finally, we discuss the implications for the quantification of entanglement between two fermionic modes in Sec. VI, before we investigate situations beyond two modes in Sec. VII.

* pmxnf@nottingham.ac.uk

† pmxal3@nottingham.ac.uk

‡ david.edward.bruschi@gmail.com

II. THE FERMIONIC FOCK SPACE

Let us consider a (discrete) set of solutions ψ_n to a (relativistic) field equation, e.g., the Dirac equation. In the decomposition of the quantum field ψ each mode function ψ_n is assigned an annihilation operator b_n and a creation operator b_n^\dagger , which satisfy the canonical anti-commutation relations

$$\{b_m, b_n^\dagger\} = \delta_{mn}, \quad (1a)$$

$$\{b_m, b_n\} = \{b_m^\dagger, b_n^\dagger\} = 0, \quad (1b)$$

where $\{\cdot, \cdot\}$ denotes the anti-commutator. One may introduce a different notation for the annihilation and creation operators for modes with opposite charge, see, e.g., Ref. [8], but for the purpose of our analysis here this is inconsequential and we can work only with b_n and b_n^\dagger . The creation operators b_n^\dagger , acting upon the vacuum state $|0\rangle$, will populate the vacuum with a single excitation, i.e.,

$$|\psi_n\rangle = b_n^\dagger |0\rangle, \quad (2)$$

while the vacuum is annihilated by all b_n , i.e., $b_n |0\rangle = 0 \forall n$. As can be quickly seen from this property and Eq. (1a), the states $|\psi_n\rangle$ are orthonormal. The states $|\psi_n\rangle$ further form a complete basis of the single particle Hilbert space \mathcal{H}_{1-p} , whereas $|0\rangle \in \mathcal{H}_{0-p} \neq \mathcal{H}_{1-p}$. A general state in \mathcal{H}_{1-p} has the form

$$|\psi^{1-p}\rangle = \sum_i \mu_i |\psi_i\rangle, \quad (3)$$

with $\sum_i |\mu_i|^2 = 1$ such that $\langle \psi^{1-p} | \psi^{1-p} \rangle = 1$. Let us now turn to states of multiple fermions. A second fermion can be added to the state (2) by the action of another creation operator b_m^\dagger , i.e.,

$$b_m^\dagger b_n^\dagger |0\rangle \propto |\psi_m, \psi_n\rangle. \quad (4)$$

Clearly, the anticommutation relations (1) require the two-fermion state to be antisymmetric with respect to the exchange of the mode labels m and n . We therefore define

$$\begin{aligned} |\psi_m, \psi_n\rangle &= b_m^\dagger b_n^\dagger |0\rangle = |\psi_m\rangle \wedge |\psi_n\rangle \\ &= \frac{1}{\sqrt{2}} (|\psi_m\rangle \otimes |\psi_n\rangle - |\psi_n\rangle \otimes |\psi_m\rangle). \end{aligned} \quad (5)$$

The two-fermion states are thus elements of the anti-symmetrized tensor product space of two single-fermion Hilbert spaces, i.e.,

$$\mathcal{H}_{2-p} = \bar{S}(\mathcal{H}_{1-p} \otimes \mathcal{H}_{1-p}), \quad (6)$$

and a general state within this space can be written as

$$|\psi^{2-p}\rangle = \sum_{i,j} \mu_{ij} |\psi_i, \psi_j\rangle, \quad (7)$$

where the coefficients μ_{ij} form an antisymmetric matrix. States with more than two fermions can then be constructed by anti-symmetrizing over the corresponding number of single-fermion states. Finally, the n -mode *fermionic Fock space* $\bar{\mathcal{F}}_n$ is simply given as the direct sum over all fermion numbers of the anti-symmetrized Hilbert spaces, i.e.,

$$\begin{aligned} \bar{\mathcal{F}}_n(\mathcal{H}_{1-p}) &= \bigoplus_{m=1}^n \bar{S}(\mathcal{H}_{1-p}^{\otimes m}) \\ &= \mathcal{H}_{0-p} \oplus \mathcal{H}_{1-p} \oplus \bar{S}(\mathcal{H}_{1-p} \otimes \mathcal{H}_{1-p}) \oplus \dots, \end{aligned} \quad (8)$$

where $\mathcal{H}^{\otimes m}$ denotes the m -fold tensor product and we write \mathcal{H}_{0-p} as $\mathcal{H}_{1-p}^{\otimes 0}$. A general state in the space $\bar{\mathcal{F}}_n$ can be written as

$$\begin{aligned} |\Psi^{\bar{\mathcal{F}}_n}\rangle &= \mu_0 |0\rangle \oplus \sum_{i=1}^n \mu_i |\psi_i\rangle \\ &\oplus \frac{1}{\sqrt{2}} \sum_{j,k} \mu_{jk} |\psi_j\rangle \wedge |\psi_k\rangle \oplus \dots \end{aligned} \quad (9)$$

Let us now simplify the notation. From now on we will denote states in the fermionic Fock space by double-lined Dirac notation, i.e., $\|\cdot\>\$ instead of $|\cdot\rangle$, where the antisymmetric “wedge” product is implied when two vectors are multiplied, i.e., $\|\cdot\>\|\cdot\>\ = \|\cdot\>\wedge\|\cdot\>\$. Furthermore, let us use the common “occupation number” notation and write 1_n instead of ψ_n to denote an excitation in the mode n . Finally, we omit the symbol for the direct sum and simply keep in mind that states with different numbers of excitations occupy different sectors of the fermionic Fock space. With this convention in mind we can rewrite Eq. (9) as

$$\begin{aligned} \|\Psi\>\ = & \mu_0 \|0\>\ + \sum_{i=1}^n \mu_i \|1_i\>\ \\ & + \frac{1}{\sqrt{2}} \sum_{j,k} \mu_{jk} \|1_j\>\|1_k\>\ + \dots \end{aligned} \quad (10)$$

For the adjoint space we use the convention (compare to Eq. (5))

$$\begin{aligned} \langle\langle 1_n \| \langle\langle 1_m \| &:= \langle\langle 0 \| b_n b_m = (b_m^\dagger b_n^\dagger \|0\>\rangle^\dagger \\ &= -\frac{1}{\sqrt{2}} (\langle\psi_n| \otimes \langle\psi_m| - \langle\psi_m| \otimes \langle\psi_n|), \end{aligned} \quad (11)$$

which allows us to write

$$\langle\langle 1_m \| \langle\langle 1_n \| 1_i \>\|1_j\>\ = \delta_{ni} \delta_{mj} - \delta_{nj} \delta_{mi}. \quad (12)$$

This notation is more convenient for computations in the fermionic Fock space. It should be noted that, in standard quantum information notation, the position of a “ket” corresponds to a particular ordering of the

subspaces with respect to the tensor product structure of the total space. Here, however, there is no tensor product structure corresponding to different modes according to which the vectors $\|\cdot\rangle\rangle$ can be naturally ordered.

III. DENSITY OPERATORS IN THE FERMIONIC FOCK SPACE

In complete analogy to the usual case of mixed states on tensor product spaces we can now construct incoherent mixtures of pure state in a fermionic Fock space. Let us first consider the projector on the state $\|\Psi\rangle\rangle$ from Eq. (10), i.e.,

$$\begin{aligned} \|\Psi\rangle\rangle\langle\langle\Psi| &= |\mu_0|^2 \|0\rangle\rangle\langle\langle 0| + \sum_{i,i'} \mu_i \mu_{i'}^* \|1_i\rangle\rangle\langle\langle 1_{i'}| \\ &+ \sum_{j,j',k,k'} \mu_{jk} \mu_{j'k'}^* \|1_j\rangle\rangle\langle\langle 1_k| \langle\langle 1_{j'}| \langle\langle 1_{k'}| \\ &+ \sum_i \left(\mu_i \mu_0^* \|1_i\rangle\rangle\langle\langle 0| + \text{h.c.} \right) + \dots \end{aligned} \quad (13)$$

Let us check that such an object satisfies the criteria for a density operator:

- (i) It can be immediately noticed that (13) provides a *hermitean* operator.
- (ii) The *normalization*, i.e., $\text{Tr}(\|\Psi\rangle\rangle\langle\langle\Psi|) = 1$, is guaranteed by the normalization of $\|\Psi\rangle\rangle$. In other words, the trace of (13) is well defined and independent of the chosen (complete, orthonormal) basis in $\bar{\mathcal{F}}$.
- (iii) *Positivity*: Finally, the eigenvalues of $\|\Psi\rangle\rangle\langle\langle\Psi|$ are well defined, i.e., (13) can be represented as a diagonal matrix with diagonal entries $\{1, 0, 0, \dots\}$, which clearly is a positive semi-definite spectrum.

We can then simply form incoherent mixtures of such pure states using convex sums, i.e.,

$$\varrho = \sum_n p_n \|\Psi_n\rangle\rangle\langle\langle\Psi_n| \quad (14)$$

where $\sum_n p_n = 1$, to construct the elements of the *Hilbert-Schmidt space* $\mathcal{H}_S(\bar{\mathcal{F}})$ over the fermionic Fock space. Properties (i) and (ii) can trivially be seen to be satisfied for such *mixed* states. The positivity of (14), however, requires some additional comments. The operator ϱ can be diagonalized by a unitary transformation U on $\bar{\mathcal{F}}$, which in turn can be constructed from exponentiation of hermitean or anti-hermitean operators formed from algebra elements b_n and b_m^\dagger . Operationally this procedure is rather elaborate. A simpler approach is the diagonalization a matrix representation of ϱ . As

we shall see in Sec. IV, the matrix representation of ϱ is not unique, but all possible representations $\pi_i(\varrho)$ are unitarily equivalent, such that their eigenvalues all coincide with those of ϱ , i.e.,

$$\text{spectr}(\pi_i(\varrho)) = \text{spectr}(\varrho) \quad \forall i. \quad (15)$$

IV. THE FERMIONIC AMBIGUITY

Let us now turn to the apparent ambiguity in such fermionic systems when quantum information tasks are considered. It was pointed out in Ref. [28] that the anti-commutation relations (1) do not suggest a natural choice for the basis vectors of the fermionic Fock space for the multi-particle sector, i.e., for two fermions in the modes m and n , either

$$\|1_m\rangle\rangle\langle\langle 1_n| \quad \text{or} \quad \|1_n\rangle\rangle\langle\langle 1_m| = -\|1_m\rangle\rangle\langle\langle 1_n| \quad (16)$$

could be used to represent the physical state. This becomes of importance when we try to map the states in a fermionic n -mode Fock space to vectors in a n -fold tensor product space, i.e.,

$$\pi_i : \bar{\mathcal{F}}_n \longrightarrow \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n \quad (17a)$$

$$\|\psi\rangle\rangle \xrightarrow{\pi_i} |\psi_{(i)}\rangle \quad (17b)$$

$$\varrho \xrightarrow{\pi_i} \pi_i(\varrho) \quad (17c)$$

where the spaces $\mathcal{H}_i = \mathbb{C}^2$ ($i = 1, \dots, n$) are identical, single-qubit Hilbert spaces. The mappings π_i are unitary, i.e., $\langle\langle\phi|\psi\rangle\rangle = \langle\phi_{(i)}|\psi_{(i)}\rangle$ and $\text{Tr}(\varrho\sigma) = \text{Tr}(\pi_i(\varrho)\pi_i(\sigma))$. This implies that the maps π_i for different i are unitarily equivalent. In particular, the different matrix representations $\pi_i(\varrho)$ are related by multiplication of selected rows and columns of the matrix by (-1) .

In the language of quantum information theory the states $\psi_{(i)}$ are related by *global unitary* transformations. It thus becomes apparent that the entanglement of $\pi_i(\varrho)$ with respect to a bipartition

$$\mathcal{H}_{\mu_1} \otimes \dots \otimes \mathcal{H}_{\mu_m} | \mathcal{H}_{\mu_{m+1}} \otimes \dots \otimes \mathcal{H}_{\mu_n} \quad (18)$$

will generally depend on the chosen mapping.

Clearly, this is an unfavorable situation, but the inequivalence of entanglement measures for different such mappings has been noted before, see, e.g., Refs. [13, 23, 29], while other investigations [8–10] did not suffer from any problems due to this ambiguity. Recently, the authors of Ref. [29] suggested that the ambiguity can be resolved by restrictions imposed by charge superselection rules, while Ref. [11] suggested a solution by enforcing a particular operator ordering. We will discuss both of these approaches in Sec. V, where we present simple and physically intuitive criteria for quantum information processing on a fermionic Fock space. Most

importantly, we will show in Secs. V and VII that mappings of the type of (17) can only be considered to be consistent when limiting the analysis to two fermionic modes obeying charge superselection, but not beyond this regime.

V. THE PARTIAL TRACE AMBIGUITY

While the sign ambiguity in the sense of the different mappings π_i is the superficial cause of the issue we want to discuss now a separate, and in some sense more fundamental problem: partial traces over “mode subspaces”. We are interested in the entanglement between modes of a fermionic quantum field. However, in the structure of the Fock space, there is no tensor product decomposition into Hilbert spaces for particular modes, see, e.g., Eq. (5). Only a tensor product structure with respect to individual fermions is available, but, since the particles are indistinguishable the entanglement between two particles in this sense has to be defined very carefully [20]. This issue is not unique for fermions and is sometimes referred to as “fluffy bunny” entanglement, see Ref. [21].

For the decomposition into different modes we only have a wedge product structure available. In Ref. [29] the authors suggest that entanglement should be considered with respect to this special case of the “braided tensor product”. As far as the construction of the density operators with respect to such a structure is concerned we agree with this view, see Sec. III, and no ambiguities arise regarding the description of the total n -mode system. However, the crucial problem lies in the definition of the partial tracing over a subset of the n modes. This is best illustrated for a simple example: Consider a system of two fermionic modes labelled κ and κ' . A general, mixed state of these two modes can be written as

$$\begin{aligned} \varrho_{\kappa\kappa'} = & \alpha_1 \|0\rangle\langle 0| + \alpha_2 \|1_{\kappa'}\rangle\langle 1_{\kappa'}| \\ & + \alpha_3 \|1_{\kappa}\rangle\langle 1_{\kappa}| + \alpha_4 \|1_{\kappa}\rangle\langle 1_{\kappa'}| \\ & + \left(\beta_1 \|0\rangle\langle 1_{\kappa'}| + \beta_2 \|0\rangle\langle 1_{\kappa}| \right. \\ & + \beta_3 \|0\rangle\langle 1_{\kappa'}| + \beta_4 \|1_{\kappa'}\rangle\langle 1_{\kappa}| \\ & + \beta_5 \|1_{\kappa'}\rangle\langle 1_{\kappa'}| + \beta_6 \|1_{\kappa}\rangle\langle 1_{\kappa'}| \\ & \left. + h.c. \right) \end{aligned} \quad (19)$$

where appropriate restrictions on the coefficients $\alpha_i \in \mathbb{R}$ and $\beta_j \in \mathbb{C}$ apply, to ensure the positivity and normalization of $\varrho_{\kappa\kappa'}$. Here we have, for now, disregarded superselection rules. Let us now determine the corresponding reduced density operators (on the Fock space) for the individual modes κ and κ' . Usually one would

select a basis of the subsystem that is being traced over, e.g., for tracing over mode κ' one could choose $\{|0\rangle, |1_{\kappa'}\rangle\}$. This clearly cannot work since basis vectors with different number of excitations are orthogonal. We thus have to define the partial trace in a different way. This is equally true for bosonic fields as well. However, in contrast to the fermionic case, no ambiguities arise in such a redefinition for bosonic fields. For the diagonal elements of the reduced fermionic states the redefinition of the partial trace is straightforward as well. These elements are obtained from

$$\text{Tr}_m(\|0\rangle\langle 0|) := \|0\rangle\langle 0|, \quad (20a)$$

$$\begin{aligned} \text{Tr}_m(\|1_n\rangle\langle 1_n|) &:= \delta_{mn} \|1_n\rangle\langle 1_n| \\ &+ (1 - \delta_{mn}) \|0\rangle\langle 0|, \end{aligned} \quad (20b)$$

$$\text{Tr}_m(\|1_m\rangle\langle 1_n|) := \|1_n\rangle\langle 1_n| \quad (m \neq n), \quad (20c)$$

where $n, m = \kappa, \kappa'$. While the diagonal elements are unproblematic and do not suffer from any ambiguities, we have to be more careful with the off-diagonal elements. Three of these will not contribute, i.e.,

$$\begin{aligned} \text{Tr}_m(\|1_m\rangle\langle 1_n|) &= \text{Tr}_m(\|0\rangle\langle 1_m| \|1_n\rangle) \\ &= \text{Tr}_m(\|1_n\rangle\langle 1_m| \|1_n\rangle) = 0, \end{aligned} \quad (21)$$

and two more are unproblematic as well, i.e.,

$$\text{Tr}_m(\|0\rangle\langle 1_n|) := (1 - \delta_{mn}) \|0\rangle\langle 1_n|. \quad (22)$$

The last element,

$$\begin{aligned} \text{Tr}_m(\|1_m\rangle\langle 1_m| \|1_n\rangle) &= -\text{Tr}_m(\|1_m\rangle\langle 1_n| \|1_m\rangle) \\ &= \pm \|0\rangle\langle 1_n|, \end{aligned} \quad (23)$$

however, presents an ambiguity. If a mapping π_i to a two-qubit Hilbert space is performed, the choice of map will determine the corresponding sign in the partial trace over either of the qubits. The differences in entanglement related to the fact that $\pi_i(\varrho)$ and $\pi_j(\varrho)$ are related by a global unitary are thus explained by the relative sign between the contributions of Eq. (22) and Eq. (23) to the same element of the reduced density matrix.

However, simple *physical requirements* restrict the choice in this relative sign. Any reduced state formalism has to satisfy the simple criterion that the reduced density operator contains all the information about the subsystem that can be obtained from the global state when measurements are performed only on the respective subsystem alone.

Let us put this statement in more mathematical terms. For any bipartition $A|B$ of a Hilbert space \mathcal{H} (with respect to any braided tensor product structure on \mathcal{H})

and any state $\rho \in \mathcal{H}$ the partial trace operation Tr_B must satisfy

$$\langle \mathcal{O}_n(A) \rangle_\rho = \langle \mathcal{O}_n(A) \rangle_{\text{Tr}_B(\rho)}, \quad (24)$$

where $\langle \mathcal{O} \rangle_\rho$ denotes the expectation value of the operator \mathcal{O} in the state ρ and $\{\mathcal{O}_n(A)\}$ is the set of all (hermitean) operators that act on the subspace A only. For the operator $\varrho_{\kappa\kappa'}$ from Eq. (19) the condition (24) can be written as

$$\text{Tr}(\mathcal{O}_n(\kappa)\varrho_{\kappa\kappa'}) = \text{Tr}(\mathcal{O}_n(\kappa)\varrho_\kappa), \quad (25)$$

where $\varrho_\kappa = \text{Tr}_{\kappa'}(\varrho_{\kappa\kappa'})$. This consistency condition uniquely determines the relative signs between different contributions to the same elements of ϱ_κ . Let us consider the (hermitean) operators $(b_\kappa + b_\kappa^\dagger)$ and $i(b_\kappa - b_\kappa^\dagger)$. Their expectation values for the global state $\varrho_{\kappa\kappa'}$ are given by

$$\text{Tr}((b_\kappa + b_\kappa^\dagger)\varrho_{\kappa\kappa'}) = 2\text{Re}(\beta_2 + \beta_5), \quad (26a)$$

$$\text{Tr}(i(b_\kappa - b_\kappa^\dagger)\varrho_{\kappa\kappa'}) = 2\text{Im}(\beta_2 + \beta_5). \quad (26b)$$

For the mode κ' on the other hand we compute

$$\text{Tr}((b_{\kappa'} + b_{\kappa'}^\dagger)\varrho_{\kappa\kappa'}) = 2\text{Re}(\beta_1 - \beta_6), \quad (27a)$$

$$\text{Tr}(i(b_{\kappa'} - b_{\kappa'}^\dagger)\varrho_{\kappa\kappa'}) = 2\text{Im}(\beta_1 - \beta_6). \quad (27b)$$

Equations (26) and (27) determine the sign in Eq. (23) and we find the reduced states

$$\varrho_\kappa = \text{Tr}_{\kappa'}(\varrho_{\kappa\kappa'}) = (\alpha_1 + \alpha_2) \parallel 0 \rangle \langle 0 \parallel \quad (28a)$$

$$+ (\alpha_3 + \alpha_4) \parallel 1_\kappa \rangle \langle 1_\kappa \parallel \\ + ((\beta_2 + \beta_5) \parallel 0 \rangle \langle 1_\kappa \parallel + \text{h.c.})$$

$$\varrho_{\kappa'} = \text{Tr}_\kappa(\varrho_{\kappa\kappa'}) = (\alpha_1 + \alpha_3) \parallel 0 \rangle \langle 0 \parallel \quad (28b)$$

$$+ (\alpha_2 + \alpha_4) \parallel 1_{\kappa'} \rangle \langle 1_{\kappa'} \parallel \\ + ((\beta_1 - \beta_6) \parallel 0 \rangle \langle 1_{\kappa'} \parallel + \text{h.c.})$$

for the modes κ and κ' respectively. Notice that this formally corresponds to tracing “inside-out”, that is, first (anti-)commuting operators towards the projector on the vacuum state before removing them, such that

$$\text{Tr}_m(b_m^\dagger \parallel 0 \rangle \langle 0 \parallel b_m b_n) = \parallel 0 \rangle \langle 1_n \parallel. \quad (29)$$

We have now arrived at a point where we can make a general statement about the consistency conditions. Let us formulate this in the following theorem.

Theorem 1. *Given a density operator $\varrho_{1,\dots,n} \in \mathcal{H}_S(\bar{\mathcal{F}}_n)$ for n fermionic modes (labelled $1, \dots, n$) the consistency conditions (24) completely determine the reduced states on $\mathcal{H}_S(\bar{\mathcal{F}}_m)$ for any m with $1 < m < n$.*

Proof. This can be seen in the following way: for any matrix element

$$\gamma b_{\mu_1}^\dagger \dots b_{\mu_i}^\dagger \parallel 0 \rangle \langle 0 \parallel b_{\nu_1} \dots b_{\nu_j} \quad (30)$$

of an $(n-1)$ -mode reduced state $\varrho_{1,\dots,(n-1)} = \text{Tr}_n(\varrho_{1,\dots,n})$, where $\gamma \in \mathbb{C}$ and the sets

$$\mu := \{\mu_1, \dots, \mu_i\} \subseteq \{1, 2, \dots, (n-1)\} \quad (31a)$$

$$\text{and } \nu := \{\nu_1, \dots, \nu_j\} \subseteq \{1, 2, \dots, (n-1)\} \quad (31b)$$

label subsets of the mode operators for the $(n-1)$ modes, can have contributions from at most two matrix elements of $\varrho_{1,\dots,n}$, i.e.

$$\text{Tr}_n(\gamma_0 b_{\mu_1}^\dagger \dots b_{\mu_i}^\dagger \parallel 0 \rangle \langle 0 \parallel b_{\nu_1} \dots b_{\nu_j}), \quad (32a)$$

$$\text{and } \text{Tr}_n(\gamma_1 b_{\mu_1}^\dagger \dots b_{\mu_i}^\dagger b_n^\dagger \parallel 0 \rangle \langle 0 \parallel b_n b_{\nu_1} \dots b_{\nu_j}). \quad (32b)$$

The composition of γ into $\gamma_1 \in \mathbb{C}$ and $\gamma_2 \in \mathbb{C}$, i.e., $\gamma = \gamma_0 \pm \gamma_1$, is determined by the consistency conditions of Eq. (24). For every matrix element (30) with corresponding partial trace contributions from (32) there exists a pair of hermitean operators

$$\mathcal{O}_x(\lambda, \tau) = b_{\lambda_1} \dots b_{\lambda_k} b_{\tau_1}^\dagger \dots b_{\tau_l}^\dagger \quad (33a)$$

$$+ b_{\tau_1} \dots b_{\tau_l} b_{\lambda_k}^\dagger \dots b_{\lambda_1}^\dagger,$$

$$\mathcal{O}_p(\lambda, \tau) = b_{\lambda_1} \dots b_{\lambda_k} b_{\tau_1}^\dagger \dots b_{\tau_l}^\dagger \quad (33b)$$

$$- i b_{\tau_1} \dots b_{\tau_l} b_{\lambda_k}^\dagger \dots b_{\lambda_1}^\dagger,$$

with $\lambda := \{\lambda_1, \dots, \lambda_k\} = \mu/\nu$ and $\tau := \{\tau_1, \dots, \tau_l\} = \nu/\mu$, that uniquely determine the relative sign of γ_1 and γ_2 . These operators are unique up to an overall multiplication with scalars. The tracing procedure can be repeated when any other of the $(n-1)$ remaining modes are traced over. Since the order of the partial traces is of no importance for the final reduced state, all reduced density operators are completely determined. \square

Consequently, the reduced density matrices in the fermionic Fock space can be considered as proper density operators, i.e., they are hermitean, normalized and their eigenvalues are well defined and non-negative. Moreover, since the eigenvalues are free of ambiguities, all functions of these eigenvalues, in particular all entropy measures for density operators, are well defined. Let us stress here that this analysis does not depend on any superselection rules that might be imposed in addition. We will see how these enter the problem when mappings to qubits are attempted in Sec. VI.

VI. ENTANGLEMENT OF FERMIONIC MODES

We are now in a position to reconsider a measure of entanglement between fermionic modes. We can de-

fine the entanglement of formation \bar{E}_{oF} for fermionic systems with respect to a chosen bipartition $A|B$ as

$$\bar{E}_{oF}(\varrho) = \min_{\{p_n, \|\Psi_n\|\}} \sum_n p_n \mathcal{E}(\|\Psi_n\|), \quad (34)$$

in complete analogy to the usual definition [32]. Here the minimum is taken over all pure state ensembles $\|\Psi_n\|$ that realize ϱ according to Eq. (14) and $\mathcal{E}(\|\Psi\|)$ denotes the entropy of entanglement of the pure state $\|\Psi\|$. Since the entropy of entanglement, e.g., using the von-Neumann entropy, is a function of the eigenvalues of the reduced states $\text{Tr}_B(\|\Psi\|\langle\Psi\|)$ or $\text{Tr}_A(\|\Psi\|\langle\Psi\|)$ alone, we can conclude that this is a well defined quantity. As pointed out in Ref. [23], the minimization in Eq. (34) can be restricted to pure state decompositions that respect superselection rules. Since this restriction limits the set of states over which the minimization is carried out, the quantity without this restriction will be a lower bound to the “physical” entanglement of formation. For two fermionic modes the minimization over all states that respect superselection rules can indeed be carried out, see Ref. [23]. However, in general this step will be problematic.

Let us now turn to some operational entanglement measure, in particular, let us investigate if and how the *negativity* \mathcal{N} (see, e.g., Ref. [33]) and the *concurrence* C (see, e.g., Ref. [32]) can be computed to quantify fermionic mode entanglement. Both of these measures are operationally based on the tensor product structure of qubits. We will here define the negativity as

$$\mathcal{N} := \sum_i \frac{(\lambda_i - |\lambda_i|)}{2}, \quad (35)$$

where λ_i are the eigenvalues of the partially transposed density matrix. The partial transposition is a map that is well defined only for basis vectors on a tensor product space. To employ this measure, let us therefore try to find a mapping π_i that is consistent with the conditions of Eq. (24). Starting with the two-mode state $\varrho_{\kappa\kappa'}$ of Eq. (19) we are looking for a map π that takes $\{\|0\|, \|1_\kappa\|, \|1_{\kappa'}\|, \|1_\kappa\| \|1_{\kappa'}\|\}$ to $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where $|mn\rangle = |m\rangle \otimes |n\rangle \in \mathcal{H}_\kappa \otimes \mathcal{H}_{\kappa'}$, such that

$$\varrho \mapsto \pi(\varrho), \quad (36a)$$

$$\varrho_\kappa \mapsto \pi(\varrho_\kappa) = \text{Tr}_{\kappa'}(\pi(\varrho)), \quad (36b)$$

$$\varrho_{\kappa'} \mapsto \pi(\varrho_{\kappa'}) = \text{Tr}_\kappa(\pi(\varrho)). \quad (36c)$$

A graphical representation of the consistency condition is shown in Fig. 1. It is quite simple to check that these requirements generally cannot be met, i.e., writing $\varrho_{\kappa\kappa'}$ of Eq. (19) as a matrix with respect to the

basis $\{\|0\|, \|1_\kappa\|, \|1_{\kappa'}\|, \|1_\kappa\| \|1_{\kappa'}\|\}$ we get

$$\varrho = \begin{pmatrix} \alpha_1 & \beta_1 & \beta_2 & \beta_3 \\ \beta_1^* & \alpha_2 & \beta_4 & \beta_5 \\ \beta_2^* & \beta_4^* & \alpha_3 & \beta_6 \\ \beta_3^* & \beta_5^* & \beta_6^* & \alpha_4 \end{pmatrix}. \quad (37)$$

A mapping of the desired type should be obtained by

$$\begin{array}{ccc} \varrho_{\kappa\kappa'} & \xrightarrow{\pi} & \pi(\varrho_{\kappa\kappa'}) \\ \text{Tr}_{\kappa'} \downarrow & & \downarrow \text{Tr}_{\kappa'} \\ \varrho_\kappa & \xrightarrow{\pi} & \pi(\varrho_\kappa) \end{array}$$

FIG. 1. Graphical representation of consistent mapping: A mapping $\pi : \varrho \mapsto \pi(\varrho)$ from the space $\mathcal{H}_S(\bar{\mathcal{F}}_2)$ to $\mathcal{H}_\kappa \otimes \mathcal{H}_{\kappa'}$ is considered to be consistent if it commutes with the partial trace operation.

multiplying any number of rows and the corresponding columns by (-1) and considering the resulting matrix as the representation of $\pi(\varrho)$ on $\mathcal{H}_\kappa \otimes \mathcal{H}_{\kappa'}$. The desired result should have a relative sign switch between β_1 and β_6 , while the signs in front of β_2 and β_5 should be the same. This clearly is not possible unless some of the coefficients vanish identically, e.g., by imposing superselection rules. For example, conservation of charge would require the coefficients $\beta_1, \beta_2, \beta_5, \beta_6$, and, depending on the charge of the modes κ and κ' , either β_3 or β_4 to vanish identically. In this way only incoherent mixtures of pure states with different charge are allowed, but not coherent superpositions.

We thus find that *two fermionic modes can only be consistently represented as two qubits when charge superselection is respected*. In that case only one off-diagonal element can be non-zero and the sign of this element is insubstantial, i.e., it does not influence the reduced states or the value of any entanglement measure. In particular, the results for entanglement generation and degradation between two fermionic modes presented in Refs. [6, 8, 9] respect both charge superselection and the consistency conditions of Eq. (24).

Let us return to the choice of entanglement measure for the permitted mappings to two qubits. We now restrict the entanglement of formation \bar{E}_{oF} as defined in Eq. (34) to states that obey charge superselection, as suggested in Ref. [23]. As discussed earlier this means the usual entanglement of formation E_{oF} provides a lower bound to \bar{E}_{oF} , i.e.,

$$E_{oF} \leq \bar{E}_{oF}. \quad (38)$$

For two qubits $E_{oF} = E_{oF}(C)$ is a monotonically increasing function of the concurrence C . We propose an analogous functional dependence of $\bar{E}_{oF} = \bar{E}_{oF}(\bar{C})$ on a parameter \bar{C} , that we call “fermionic concurrence”.

Evidently, the function $\bar{C}(\rho)$ is an entanglement monotone that is bounded from below by the usual concurrence C . As shown in Ref. [34] the negativity \mathcal{N} further provides a lower bound to the concurrence, i.e., in our convention of Eq. (35), $2\mathcal{N} \leq C$. Consequently, the negativity provides a lower bound to \bar{C} , i.e.,

$$2\mathcal{N} \leq C \leq \bar{C}. \quad (39)$$

For two modes it is thus at least possible to compute lower bounds to entanglement measures explicitly. It was suggested in Ref. [20] that conventional entanglement measures overestimate the quantum correlations that can physically be extracted from fermionic systems. The operations that can be performed on each single-mode subsystem are limited by (charge) superselection as well. However, we conjecture that the inaccessible entanglement between the fermionic modes can always be swapped to two (uncharged) bosonic modes for which the local bases can be chosen arbitrarily.

VII. FERMIONIC ENTANGLEMENT BEYOND TWO MODES

Finally, let us consider the entanglement between more than two fermionic modes. In principle, any measure of entanglement that is based on entropies of the subsystems is well defined on the fermionic Fock space, as we have discussed. However, we would like to employ operational measures. Let us therefore start by attempting a consistent mapping from three fermionic modes to three qubits, in analogy to the two-mode case in Sec. VI. For simplicity we assume that the modes κ , κ' and κ'' all have equal charge such that the most general mixed state of these modes can be written as

$$\begin{aligned} \varrho_{\kappa\kappa'\kappa''} = & \mu_1 |0\rangle\langle 0| + \mu_2 |1_{\kappa''}\rangle\langle 1_{\kappa''}| \\ & + \mu_3 |1_{\kappa'}\rangle\langle 1_{\kappa'}| + \mu_4 |1_{\kappa'}\rangle\langle 1_{\kappa''}| \langle 1_{\kappa'}| \\ & + \mu_5 |1_{\kappa}\rangle\langle 1_{\kappa}| + \mu_6 |1_{\kappa}\rangle\langle 1_{\kappa'}| \langle 1_{\kappa''}| \langle 1_{\kappa}| \\ & + \mu_7 |1_{\kappa}\rangle\langle 1_{\kappa'}| \langle 1_{\kappa'}| \langle 1_{\kappa}| \\ & + \mu_8 |1_{\kappa}\rangle\langle 1_{\kappa'}| \langle 1_{\kappa''}| \langle 1_{\kappa'}| \langle 1_{\kappa}| \\ & + \left(\nu_1 |1_{\kappa''}\rangle\langle 1_{\kappa'}| + \nu_2 |1_{\kappa''}\rangle\langle 1_{\kappa}| \right. \\ & + \nu_3 |1_{\kappa'}\rangle\langle 1_{\kappa}| + \nu_4 |1_{\kappa'}\rangle\langle 1_{\kappa''}| \langle 1_{\kappa'}| \langle 1_{\kappa}| \\ & + \nu_5 |1_{\kappa'}\rangle\langle 1_{\kappa''}| \langle 1_{\kappa'}| \langle 1_{\kappa}| \\ & \left. + \nu_6 |1_{\kappa}\rangle\langle 1_{\kappa''}| \langle 1_{\kappa'}| \langle 1_{\kappa}| + h.c. \right). \end{aligned} \quad (40)$$

The relevant consistency conditions to construct the three different reduced two-mode density matrices $\varrho_{\kappa\kappa'}$,

$\varrho_{\kappa\kappa''}$ and $\varrho_{\kappa'\kappa''}$ are given by

$$\text{Tr}((b_{\kappa}^{\dagger}b_{\kappa'} + b_{\kappa'}^{\dagger}b_{\kappa})\varrho_{\kappa\kappa'\kappa''}) = 2\text{Re}(\nu_3 + \nu_4), \quad (41a)$$

$$\text{Tr}((b_{\kappa}^{\dagger}b_{\kappa''} + b_{\kappa''}^{\dagger}b_{\kappa})\varrho_{\kappa\kappa'\kappa''}) = 2\text{Re}(\nu_2 - \nu_5), \quad (41b)$$

$$\text{Tr}((b_{\kappa'}^{\dagger}b_{\kappa''} + b_{\kappa''}^{\dagger}b_{\kappa'})\varrho_{\kappa\kappa'\kappa''}) = 2\text{Re}(\nu_1 + \nu_6). \quad (41c)$$

Again, the correct partial traces are obtained by tracing “inside-out”, see Eq. (29). This is not a coincidence. The prescription for the partial trace to anti-commute operators towards the projector of the vacuum state before eliminating them takes into account the number of anti-commutations occurring in computations of the expectation values of Eq. (24). A matrix representation of the three mode state $\varrho_{\kappa\kappa'\kappa''}$ is given by

$$\varrho_{\kappa\kappa'\kappa''} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & \nu_1 & 0 & \nu_2 & 0 & 0 & 0 \\ 0 & \nu_1^* & \mu_3 & 0 & \nu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_4 & 0 & \nu_4 & \nu_5 & 0 \\ 0 & \nu_2^* & \nu_3^* & 0 & \mu_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu_4^* & 0 & \mu_6 & \nu_6 & 0 \\ 0 & 0 & 0 & \nu_5^* & 0 & \nu_6^* & \mu_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_8 \end{pmatrix}. \quad (42)$$

Similar as before one can try to interpret Eq. (42) as a matrix representation of a three qubit state and exchange the signs of the basis vectors in the three qubit state such that the consistency conditions of Eq. (41) are met, i.e., opposite signs in front of ν_2 and ν_6 , while the signs in front of the pairs ν_3, ν_4 and ν_1, ν_6 are each the same. This is not possible, even though superselection rules are respected. This suggests that the superselection rules only coincidentally aid the fermionic qubit mapping for two modes. They simply force all the problematic coefficients to disappear. However, for more than two modes we find here that a mapping to a tensor product space cannot be performed consistently in general. Therefore, computing a measure like the negativity to determine the entanglement between more than two modes appears to be meaningless. Due to the lack of practical alternatives the minimization over all states consistent with charge superselection to find \bar{E}_{oF} of Eq. (34) should be considered since the restriction of the set of permissible states could make this computation feasible.

VIII. CONCLUSION

We have discussed the implementation of fermionic modes as fundamental objects for quantum information tasks. The foundation of this task is the rigorous construction of the notion of mode-subsystems in a fermionic Fock space. We have demonstrated, that this can be achieved despite the absence of simple tensor product structure. Our simple consistency conditions give a clear picture of this process, which can

operationally be easily executed by performing partial traces “inside-out”. Thus we show that fermionic mode entanglement, quantified by the (fermionic) entanglement of formation, is indeed a well defined concept, free of any ambiguities.

However, problems arise when mappings from the fermionic Fock space to qubit spaces are attempted. We have explicitly demonstrated in two examples, for two and three modes, that such mappings cannot generally succeed. Only in the limited case where only two modes are considered and the quantum states obey charge superselection can one meaningfully speak of an equivalence between the two fermionic modes and two qubits. In this case the application of tools such as the negativity or concurrence is justified. We have argued that these measures will at least provide a lower bound to genuine measures of fermionic mode entanglement.

Nonetheless, open questions remain. In particular, it is not clear if any operational measures exist for situation beyond two qubits. In Ref. [10] witnesses for genuine multipartite entanglement are employed, which are

completely compatible with the framework we have presented here, but these witnesses can only provide lower bounds on entropic entanglement measures.

Finally, we have conjectured that the entanglement in fermionic modes is accessible even in spite of superselection rules that restrict the possible operations performed on single modes by means of entanglement swapping. The investigation of this question, while beyond the scope of this article, will certainly be of future interest.

ACKNOWLEDGMENTS

We thank Samuel L. Braunstein, Ivette Fuentes, Marcus Huber, Jorma Louko, Eduardo Martín-Martínez, Miguel Montero, Carlos Sabín, Michael Skotiniotis and Vlatko Vedral for useful discussions and comments. N. F. acknowledges support from EP-SRC (CAF Grant No. EP/G00496X/2 to I. F.). We also want to thank Perimeter Institute for Theoretical Physics and the organizers of the RQI-N 2012 conference, where part of this research was conducted.

-
- [1] Y. Shi, *Phys. Rev. D* **70**, 105001 (2004).
 - [2] P. M. Alsing, I. Fuentes-Schuller, R. B. Mann and T. E. Tessier, *Phys. Rev. A* **74**, 032326 (2006).
 - [3] I. Fuentes, R. B. Mann, E. Martín-Martínez and S. Moradi, *Phys. Rev. D* **82**, 045030 (2010).
 - [4] D. E. Bruschi, J. Louko, E. Martín-Martínez, A. Dragan and I. Fuentes, *Phys. Rev. A* **82**, 042332 (2010).
 - [5] E. Martín-Martínez and I. Fuentes, *Phys. Rev. A* **83**, 052306 (2011).
 - [6] N. Friis, P. Köhler, E. Martín-Martínez and R. A. Bertlmann, *Phys. Rev. A* **84**, 062111 (2011).
 - [7] A. Smith and R. B. Mann, *Phys. Rev. A* **86**, 012306 (2012).
 - [8] N. Friis, A. R. Lee, D. E. Bruschi and J. Louko, *Phys. Rev. D* **85**, 025012 (2012).
 - [9] N. Friis, D. E. Bruschi, J. Louko and I. Fuentes, *Phys. Rev. D* **85**, 081701(R) (2012).
 - [10] N. Friis, M. Huber, I. Fuentes and D. E. Bruschi, *Phys. Rev. D* **86**, 105003 (2012).
 - [11] M. Montero and E. Martín-Martínez, *Phys. Rev. A* **85**, 024301 (2012).
 - [12] P. Zanardi, *Phys. Rev. A* **65**, 042101 (2002).
 - [13] A. Botero and B. Reznik, *Phys. Lett. A* **331**, 39-44 (2004).
 - [14] J. Schliemann, D. Loss and A. H. MacDonald, *Phys. Rev. B* **63**, 085311 (2001).
 - [15] J. Schliemann, J. I. Cirac, M. Kuś, M. Lewenstein and D. Loss, *Phys. Rev. A* **64**, 022303 (2001).
 - [16] R. Paškauskas and L. You, *Phys. Rev. A* **64**, 042310 (2001).
 - [17] Y. S. Li, B. Zeng, X. S. Liu and G. L. Long, *Phys. Rev. A* **64**, 054302 (2001).
 - [18] K. Eckert, J. Schliemann, D. Bruß and M. Lewenstein, *Annals Phys.* **299**, 88 (2002).
 - [19] Y. Shi, *Phys. Rev. A* **67**, 024301 (2003).
 - [20] H. M. Wiseman and J. A. Vaccaro, *Phys. Rev. Lett.* **91**, 097902 (2003).
 - [21] H. M. Wiseman, S. D. Bartlett and J. A. Vaccaro, e-print [arXiv:quant-ph/0309046](https://arxiv.org/abs/quant-ph/0309046) [quant-ph] (2003).
 - [22] G. C. Ghirardi and L. Marinatto, *Phys. Rev. A* **70**, 012109 (2004).
 - [23] P. Caban, K. Podlaski, J. Rembieliński, K. A. Smoliński and Z. Walczak, *J. Phys. A: Math. Gen.* **38**, L79 (2005).
 - [24] F. Iemini, R. O. Vianna, e-print [arXiv:1211.1886](https://arxiv.org/abs/1211.1886) [quant-ph] (2012).
 - [25] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
 - [26] F. Strocchi and A. S. Wightman, *J. Math. Phys.* **15**, 2198 (1974).
 - [27] G. Adesso and F. Illuminati, *Phys. Rev. A* **72**, 032334 (2005).
 - [28] M. Montero and E. Martín-Martínez, *Phys. Rev. A* **83**, 062323 (2011).
 - [29] K. Brádler and R. Jáuregui, *Phys. Rev. A* **85**, 016301 (2012).
 - [30] M. Montero and E. Martín-Martínez, *Phys. Rev. A* **85**, 016302 (2012).
 - [31] L. Heaney and V. Vedral, *Phys. Rev. Lett.* **103**, 200502 (2009).
 - [32] C. H. Bennett, D. P. Di Vincenzo, J. A. Smolin and W. K. Wootters, *Phys. Rev. A* **54**, 3824-3851 (1996).
 - [33] G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002).
 - [34] F. Verstraete, K. Audenaert, J. Dehaene and B. De Moor, *J. Phys. A: Math. Gen.* **34**, 10327 (2001).